

**Something Kinda Cool:  
Math 423, HW5, Problem 3**

**Question 3:** Let  $W_t$  be a continuous-time Brownian motion. For a twice differentiable function  $f$ , show that  $m(x, t) := \mathbb{E}[f(x + W_t)]$  solves the PDE  $\partial_t m(x, t) = \frac{1}{2} \partial_{xx} m(x, t)$  with  $m(x, 0) = f(x)$ .

**Physics of Randomness: BMs to SDEs**

One of the coolest moments in my Mathematics of Finance class was finding the bridge between pure probability theory and deterministic physical equations. We were tasked with proving that the expectation of a function applied to a Brownian motion solves a specific PDE, a result that completely changed how I view stochastic processes.

**The (Core) Solution:**

First, evaluating the initial condition is straightforward: because  $W_0 = 0$ ,  $m(x, 0) = \mathbb{E}[f(x + 0)] = f(x)$ . To construct the PDE, we define a shifted process  $X_t = x + W_t$ , meaning  $dX_t = dW_t$ . Applying Ito's formula to  $f(X_t)$  and writing in integral form gives:

$$f(x + W_t) = f(x) + \int_0^t f'(x + W_s) dW_s + \frac{1}{2} \int_0^t f''(x + W_s) ds$$

Taking the expectation of both sides simplifies things. Since the Ito integral with respect to Brownian motion is a martingale, its expectation is zero. Using Fubini's theorem to pass the expectation inside of the time integral, we get:

$$m(x, t) = f(x) + \frac{1}{2} \int_0^t \mathbb{E}[f''(x + W_s)] ds$$

Assuming we can pass the derivative operator inside the expectation, we find that  $\partial_{xx} m(x, s) = \mathbb{E}[f''(x + W_s)]$ . Subbing this into our integral gives  $m(x, t) = f(x) + \frac{1}{2} \int_0^t \partial_{xx} m(x, s) ds$ . Finally, taking the partial derivative with respect to time  $t$  using the Fundamental Theorem of Calculus gives us perfect our target PDE:  $\partial_t m = \frac{1}{2} \partial_{xx} m$ .  $\square$

**Building on this Result:**

The equation  $\partial_t m = \frac{1}{2} \partial_{xx} m$  is the Heat Equation. Proving that the exact same formula is behind the expected value of a purely random walk was something I thought was absolutely crazy. What made this even cooler was how we immediately built on it. In further problems, we generalized this concept. If we replace the simple Brownian Motion with a full diffusion process containing drift and volatility (like  $dX_t = b(t)dt + \sigma(t)dW_t$ ), the resulting expectation solves a more complex PDE:  $\partial_t m = b(t)\partial_x m + \frac{\sigma^2(t)}{2} \partial_{xx} m$ . We then used these mechanics to solve actual Stochastic Differential Equations, like the mean reverting Ornstein-Uhlenbeck process (where  $dX_t = \theta X_t dt + \sigma dW_t$ ,  $\theta < 0$ ).

**Why this interests me:**

Seeing how the Heat equation goes into the generalized PDEs used to evaluate SDEs showed me that randomness has a rigorous and, often, predictable geometry. This (basically the Feynman-Kac theorem) is what modern derivative pricing is built on, something we talked about a lot in Math 423. Recognizing that these deterministic PDEs are behind complicated and noisy SDEs solidified my desire to learn more and try to do research in applied probability. It proved that randomness could be translated into computational models using measure-theoretic math, which is honestly super cool.